On δ'-like potential scattering on star graphs

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Abstract. We discuss the potential scattering on the noncompact star graph. The Schrödinger operator with the short-range potential localizing in a neighborhood of the graph vertex is considered. We study the asymptotic behavior the corresponding scattering matrix in the zero-range limit. It has been known for a long time that in dimension 1 there is no non-trivial Hamiltonian with the distributional potential δ', i.e., the δ' potential acts as a totally reflecting wall. Several authors have, in recent years, studied the scattering properties of the regularizing potentials $\alpha\varepsilon^{-2}Q(x/\varepsilon)$ approximating the first derivative of the Dirac delta function. A non-zero transmission through the regularized potential has been shown to exist as $\varepsilon \to 0$. We extend these results to star graphs with the point interaction, which is an analogue of δ' potential on the line. We prove that generically such a potential on the graph is opaque. We also show that there exists a countable set of resonant intensities for which a partial transmission through the potential occurs. This set of resonances is referred to as the resonant set and is determined as the spectrum of an auxiliary Sturm-Liouville problem associated with Q on the graph.

1. Introduction

Schrödinger operators with potentials supported on a discrete set of points (such potentials are usually termed "point interactions") have attracted considerable attention both in the physical and mathematical literature over several past decades. One of the reasons for this is that such singular Hamiltonians are widely used in various application to atomic, nuclear, and solid state physics. Applications also arise in optics, for instance, in dielectric media where electromagnetic waves scatter at boundaries or thin layers. Another reason is that Schrödinger operators with point interactions often form "solvable" models in the sense that the resolvents of such operators can explicitly be calculated. Consequently the spectrum, the eigenfunctions, as well as resonances and scattering quantities, can also be determined explicitly (see [2,3] and references therein). In the physically oriented literature point interactions are often understood as sharply localized potentials, exhibiting a number of interesting features and unusual effects not seen for regular potentials.

Currently, there is increasing interest in solvable models on graphs in particular, as a reaction to a great deal of progress in fabricating graph-like structures of a semiconductor material, for which graph Hamiltonians represent a natural model (see the survey [20] for details). The idea to investigate quantum mechanics of particles confined to a graph originated with the study of free electron models of organic molecules [24–26]. Among the systems that were successfully modeled by quantum graphs we mention e.g., single-mode acoustic and electro-magnetic waveguide networks [14], Anderson transition [4] and quantum Hall systems with long range potential [6], fracton excitations in fractal structures [5], and mesoscopic quantum systems [19]. The essential component of such graph models is the wavefunction coupling in the vertices. The interface conditions have to be chosen to make the Hamiltonian self-adjoint, or in physical terms, to ensure conservation of the probability current at the vertex.

One of the most natural way to define a graph Hamiltonian corresponding to a point interaction supported by the branching point is to choose an appropriate self-adjoint extension of the corresponding free Hamiltonian with the interactions points removed. This approach has been realized in [13], where the authors found that a vertex joining n graph edges can be described by n^2 real parameters defining the interface condition at the vertex. A general form of such a coupling was described in [18] by a pair of $n \times n$ matrices A, B such that $\operatorname{rank}(A, B) = n$ and AB^* is self-adjoint; the boundary values have to satisfy the conditions

$$A\Psi(a) + B\Psi'(a) = 0. \tag{1}$$

Here the symbol $\Psi(a)$ is used for the column vector of the boundary values at the vertex a, and analogously $\Psi'(a)$ stands for the vector of the first derivatives, taken all in the outgoing direction.

Another possible approach to define the vertex coupling is to approximate a quantum graph with a thin quantum waveguide (see [1,8,12,23] and references therein). Different vertex couplings were also discovered in [7,10,11,13].

The Schrödinger operators with the Dirac delta-function and its derivatives in potentials have been studying intensively since the eighties of last century (see [2,3,9] and the references given there). One of the first well-studied Hamiltonian was the operator with Dirac's delta-function potential δ. A case of special interest arises when the δ-potential is replaced by the derivative δ' of the Dirac delta-function. Šeba [27] appears to have been among the first to consider such a Hamiltonian. To define the Hamiltonian $-\frac{d^2}{dx^2} + \delta'(x)$ he approximated δ' by regular potentials $\varepsilon^{-2}Q(x/\varepsilon)$ and then investigated the convergence of the corresponding family of regular Schrödinger operators

$$S_\varepsilon = -\frac{d^2}{dx^2} + \frac{1}{\varepsilon^2}Q\left(\frac{x}{\varepsilon}\right).$$

Here Q is an integrable function. If Q has zero mean and the integral $\int_{\mathbb{R}} t\, Q(t)\, dt$ is equal to -1, then the sequence $\varepsilon^{-2}Q(x/\varepsilon)$ converges in the sense of distributions as $\varepsilon \to 0$ to δ', which served as motivation for such an approximation. Šeba claimed that the operators S_ε converge in the uniform resolvent sense to the direct sum of the unperturbed half-line

Schrödinger operators subject to the Dirichlet boundary conditions at $x = 0$. From the viewpoint of scattering theory it means that the δ'-barrier is completely opaque, i.e., in the limit $\varepsilon \to 0$ the potential $\varepsilon^{-2}Q(x/\varepsilon)$ becomes a totally reflecting wall at $x = 0$ splitting the system into two independent subsystems lying on the half-lines $(-\infty, 0)$ and $(0, \infty)$.

But this result is in conflict with conclusions reached in [28], where the resonances in the transmission probability for δ'-like potential have been observed. In that paper an exactly solvable model with a step function αQ was considered. The authors found a discrete set of intensities α_n, for which partial transmission through the limiting δ'-potential occurs. The values α_n are roots of a transcendent equation depending on the regularization Q. Exactly solvable models with other piecewise constant potentials as well as nonrectangular regularizations of δ' have later been studied in [29–32] and the same conclusion has been drawn.

In [15, 16] the authors approximated the formal Hamiltonian $-\frac{d^2}{dx^2} + q(x) + \alpha\delta'(x)$ on the line by regular Schrödinger operators

$$S_\varepsilon(\alpha, Q) = -\frac{d^2}{dx^2} + q(x) + \frac{\alpha}{\varepsilon^2}Q\left(\frac{x}{\varepsilon}\right)$$

and the similar resonant effect was discovered. Here the function $Q \in C_0^\infty(-1, 1)$ has zero mean, α is a real coupling constant, and q is a real valued fixed potential tending to $+\infty$ as $|x| \to \infty$, which ensures that the spectrum of $S_\varepsilon(\alpha, Q)$ is discrete. The map assigning a selfadjoint operator $S(\alpha, Q)$ to each pair (α, Q) was constructed there. The choice of $S(\alpha, Q)$ was determined by proximity of its energy levels and pure states to those for the Hamiltonian with regularized potentials for small ε. It was established that for almost all coupling constants the operator $S(\alpha, Q)$ is just the direct sum of the Schrödinger operators with potential q on half-axes subject to the Dirichlet boundary conditions at the origin. But in the exceptional case the nontrivial coupling at the origin arises. The notion of the *resonant set* Σ_Q, which is the spectrum of the problem

$$-g'' + \alpha Q(t)g = 0, \quad t \in (-1, 1), \qquad g'(-1) = g'(1) = 0,$$

with respect to the spectral parameter α was introduced in [15, 16]. It was also constructed the *coupling function* $\theta_Q : \Sigma_Q \to \mathbb{R}$ defined via $\theta_Q(\alpha) = g_\alpha(1)/g_\alpha(-1)$, where g_α is an eigenfunction corresponding to the eigenvalue $\alpha \in \Sigma_Q$. In the case when the coupling constant α belongs to the resonance set, the δ'-barrier admits a partial transmission of a particle, and an appropriate wave function obeys the interface conditions

$$\psi(+0) = \theta_Q(\alpha)\psi(-0), \qquad \theta_Q(\alpha)\psi'(+0) = \psi'(-0).$$

Studies of [15, 16] have been continued in [17, 21]. First the findings of [28–32] were generalized in [21], where the scattering on an arbitrary potential of the form $\alpha\varepsilon^{-2}Q(x/\varepsilon)$ has been considered. It was proved that such a potential is asymptotically transparent only if a coupling constant belongs to the resonant set. Moreover, the scattering amplitude for the Hamiltonians $S_\varepsilon(\alpha, Q)$ and $-\frac{d^2}{dx^2}$ converges as $\varepsilon \to 0$ to

that for the limiting Hamiltonian $S(\alpha, Q)$. In [17] the authors have not only pointed out a mistake in [27], but also have showed that the operators $S_\varepsilon(\alpha, Q)$ converge in the uniform resolvent sense as $\varepsilon \to 0$ to $S(\alpha, Q)$. The results of [17, 21] were obtained for the special case $q = 0$; generally, the same can be derived without difficulty.

Many of point interactions on the line were extended to deal with graphs. One of the best known is the δ coupling at n edge vertex:

$$\psi_1(a) = \ldots = \psi_n(a), \qquad \sum_{l=1}^{n} \psi'_l(a) = \alpha \psi(a)$$

(see, e.g., [10, 11]). Such a model is a generalization of the Hamiltonian

$$-\frac{d^2}{dx^2} + \alpha \delta(x),$$

given by $S_{\alpha,\delta} f = -f''$ on the domain

$$\mathcal{D}(S_{\alpha,\delta}) = \{f \in H^2(\mathbb{R} \setminus \{0\}) \mid f(+0) = f(-0), \quad f'(+0) - f'(-0) = \alpha f(0)\}.$$

A similar generalization is possible for the model for Scrödinger operators with δ'-interactions, defined by $S_{\beta,\delta'} f = -f''$ on the set of functinctions

$$\mathcal{D}(S_{\beta,\delta'}) = \{f \in H^2(\mathbb{R} \setminus \{0\}) \mid f'(+0) = f'(-0), \quad f(+0) - f(-0) = \beta f'(0)\}.$$

This generalization was described in [10, 11]:

$$\psi'_1(a) = \ldots = \psi'_n(a), \qquad \sum_{l=1}^{n} \psi_l(a) = \beta \psi'(a),$$

and was called δ'_s coupling. In the following section we briefly sketch the findings of [22], where an analogue for the Hamiltonian $S_\varepsilon(\alpha, Q)$ was considered on the metric graph.
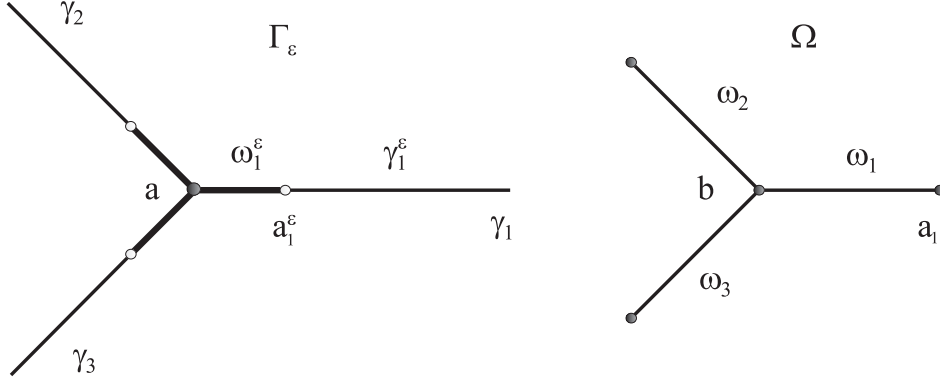
1.1. Graph Hamiltonian with the δ′-like potential

It will be convenient to recall basic notions of the theory of differential equations on graphs. By a metric graph $G = (V, E)$ we mean a finite set V of points in $\mathbb{R}^3$ (whose elements are called *vertices*) together with a set E of smooth regular curves connecting the vertices (the elements of E are called *edges*). The sets of vertices and edges of the graph G are sometimes denoted by $V(G)$ and $E(G)$ respectively. A map $f : G \to \mathbb{R}$ is said to be a *function on the graph*, and the restriction of f on the edge $g \in E(G)$ will be denoted by f_g. Let us introduce the space of functions on the graph $C^\infty(G) = \{f \mid f_g \in C^\infty(\overline{g}) \text{ for all } g \in E(G)\}$. Each edge is equipped with a natural parametrization; the differentiation is performing with respect to the natural parameter. We denote by $df/dg(a)$ the limit value of the derivative at the point $a \in V(G)$ taken in the direction away from the vertex. The integral of f over G is the sum of integrals over all edges

$$\int_G f \, dG = \sum_{g \in E(G)} \int_g f \, dg.$$

Let $L_2(G)$ be a Hilbert space with the inner product $(u, v) = \int_G u\overline{v}\,dG$ and with the norm $\|\cdot\|_{L_2(G)}$. We also introduce the Sobolev spaces $H^k(G) = \{f \in L_2(G) \mid f_g \in H^k(g)$ for all $g \in E(G)\}$ and the space of absolutely continuous functions $AC(G) = \{f \mid f_g \in AC(g)$ for all $g \in E(G)\}$.

Let us consider a noncompact star graph Γ consisting of three edges γ_1, γ_2 and γ_3. All edges are connected at the vertex a. Then $E(\Gamma) = \{\gamma_1, \gamma_2, \gamma_3\}$ and suppose that all edges are half-lines. We write a_j^ε for the point of intersection of the ε-sphere, centered at a, with the edge $\gamma_j \in E(\Gamma)$ and denote by Γ_ε a sub-partition of Γ containing new vertices a_1^ε, a_2^ε and a_3^ε. Each a_j^ε divides the edge γ_j of the graph Γ into two edges ω_j^ε and γ_j^ε of the graph Γ_ε (see figure 1). Let Ω_ε be a star subgraph of Γ_ε such that $V(\Omega_\varepsilon) = \{a, a_1^\varepsilon, a_2^\varepsilon, a_3^\varepsilon\}$ and $E(\Omega_\varepsilon) = \{\omega_1^\varepsilon, \omega_2^\varepsilon, \omega_3^\varepsilon\}$.

Figure 1. Graphs Γ_ε and Ω

By $\Omega \subset \mathbb{R}_\xi^3$ we denote the ε^{-1}-homothety of the graph Ω_ε, centered at a. Obviously, the resulting graph does not depend on the small parameter ε. The graph Ω is a star with a center at the origin b of the auxiliary space $\mathbb{R}_\xi^3$ and with the vertexes a_j, which are the images of the points a_j^ε. The edge of Ω connecting b to a_j will be denoted by ω_j.

We introduce the set $\mathcal{Q} = \{Q \in C^\infty(\Omega) \mid \int_\Omega Q\,d\Omega = 0\}$. For each nonzero element $Q \in \mathcal{Q}$ let us define the sequence

$$
Q_\varepsilon(x) = \begin{cases} \varepsilon^{-2}Q((x-a)/\varepsilon) & \text{for } x \in \Omega_\varepsilon, \\ 0 & \text{for } x \in \Gamma_\varepsilon \setminus \Omega_\varepsilon. \end{cases}
$$

The potential Q_ε is referred to as the δ'-like potential.

In [22], it was discovered the family of Schrödinger operators on the star graph

$$\mathrm{H}_\varepsilon(\alpha, Q)f = -f'' + (q + \alpha Q_\varepsilon)f,$$
$$\mathcal{D}(\mathrm{H}_\varepsilon(\alpha, Q)) = \Big\{ f \in L_2(\Gamma) \mid f, \, f' \in AC(\Gamma), \quad -f'' + (q + \alpha Q_\varepsilon)f \in L_2(\Gamma),$$

$$f_{\omega_1^\varepsilon}(a) = f_{\omega_2^\varepsilon}(a) = f_{\omega_3^\varepsilon}(a), \quad \sum_{j=1}^{3} \frac{df}{d\omega_j^\varepsilon}(a) = 0 \Big\},$$

where $Q \in \mathcal{Q}$, α is a real coupling constant, and q is a smooth real-valued potential such that $q_\gamma(x) \to +\infty$ as $|x| \to +\infty$ for all $\gamma \in E(\Gamma)$. Such behavior of q ensures

the discreteness of the spectrum of $H_\varepsilon(\alpha, Q)$. A self-adjoint operator $H(\alpha, Q)$ has been assigned to each pair (α, Q). The choice of an operator is argued by the close proximity of the energy levels and the pure states for the Hamiltonians $H_\varepsilon(\alpha, Q)$ and $H(\alpha, Q)$ respectively. Two spectral characteristics of the function Q are introduced in [22]: the *resonant set* Σ, which is the spectrum of the eigenvalue problem

$$- g'' + \alpha Q g = 0 \quad \text{on} \quad \Omega \setminus V(\Omega), \tag{2}$$

$$g_{\omega_1}(b) = g_{\omega_2}(b) = g_{\omega_3}(b), \quad \sum_{j=1}^{3} \frac{dg}{d\omega_j}(b) = 0, \tag{3}$$

$$\frac{dg}{d\omega_1}(a_1) = \frac{dg}{d\omega_2}(a_2) = \frac{dg}{d\omega_3}(a_3) = 0, \tag{4}$$

and the *coupling function* $\theta : \Sigma \to \mathbb{CP}^2$, where $\mathbb{CP}^2$ is the complex projective plane. The spectrum of the problem (2) is real, discrete and has two accumulation points $\pm\infty$. It consists of simple and double eigenvalues. Write Σ_1 for the subset of simple eigenvalues of the problem (2) and by Σ_2 we denote the subset of double eigenvalues. The resonant set can be represented as the union of Σ_1 and Σ_2.

Let α be a simple eigenvalue of the problem (2) with an eigenfunction u_α such that $u_\alpha^2(a_1) + u_\alpha^2(a_2) + u_\alpha^2(a_3) = 1$, then introduce

$$\theta_1(\alpha) = u_\alpha(a_1), \qquad \theta_2(\alpha) = u_\alpha(a_2), \qquad \theta_3(\alpha) = u_\alpha(a_3).$$

For $\alpha \in \Sigma_2$ we consider

$$\theta_1(\alpha) = v_\alpha(a_2)w_\alpha(a_3) - v_\alpha(a_3)w_\alpha(a_2),$$
$$\theta_2(\alpha) = v_\alpha(a_3)w_\alpha(a_1) - v_\alpha(a_1)w_\alpha(a_3),$$
$$\theta_3(\alpha) = v_\alpha(a_1)w_\alpha(a_2) - v_\alpha(a_2)w_\alpha(a_1),$$

where v_α and w_α form a base in the corresponding eigenspace such that $\theta_1^2(\alpha) + \theta_2^2(\alpha) + \theta_3^2(\alpha) = 1$. The coupling function is defined via $\theta(\alpha) = (\theta_1(\alpha), \theta_2(\alpha), \theta_3(\alpha))$. It if easy to check that if we change a base in the eigenspace the point $(\theta_1(\alpha), \theta_2(\alpha), \theta_3(\alpha)) \in \mathbb{C}^3$ should be replaced by $(\lambda\theta_1(\alpha), \lambda\theta_2(\alpha), \lambda\theta_3(\alpha))$ with some complex λ, hence we shall find it convenient to consider θ as a function from the resonant set into $\mathbb{CP}^2$. In the *non-resonant case*, when the coupling constant α does not belong to the resonant set, $H(\alpha, Q)$ is the direct sum of the Schrödinger operators with the potential q on edges, subject to the Dirichlet boundary conditions at the vertex a. If $\alpha \in \Sigma_1$ (the *case of simple resonance*), then the operator $H(\alpha, Q)$ acts via $H(\alpha, Q)\psi = -\psi'' + q\psi$ on an appropriate set of functions obeying the interface conditions

$$\theta_2(\alpha)\theta_3(\alpha)\psi_{\gamma_1}(a) = \theta_1(\alpha)\theta_3(\alpha)\psi_{\gamma_2}(a) = \theta_1(\alpha)\theta_2(\alpha)\psi_{\gamma_3}(a),$$
$$\theta_1(\alpha)\frac{d\psi}{d\gamma_1}(a) + \theta_2(\alpha)\frac{d\psi}{d\gamma_2}(a) + \theta_3(\alpha)\frac{d\psi}{d\gamma_3}(a) = 0 \tag{5}$$

that can be expressed by (1) with

$$A = \begin{pmatrix} \theta_3(\alpha) & 0 & -\theta_1(\alpha) \\ 0 & \theta_3(\alpha) & -\theta_2(\alpha) \\ 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \theta_1(\alpha) & \theta_2(\alpha) & \theta_3(\alpha) \end{pmatrix}.$$

In the *case of double resonance* ($\alpha \in \Sigma_2$), the interface conditions may be written as

$$\theta_2(\alpha)\theta_3(\alpha)\tfrac{d\psi}{d\gamma_1}(a) = \theta_1(\alpha)\theta_3(\alpha)\tfrac{d\psi}{d\gamma_2}(a) = \theta_1(\alpha)\theta_2(\alpha)\tfrac{d\psi}{d\gamma_3}(a),$$
$$\theta_1(\alpha)\psi_{\gamma_1}(a) + \theta_2(\alpha)\psi_{\gamma_2}(a) + \theta_3(\alpha)\psi_{\gamma_3}(a) = 0 \tag{6}$$

To derive these conditions in the form (1) we need only interchange the roles of the above matrixes A and B.

If λ is an eigenvalue of the operator $\mathrm{H}(\alpha, Q)$, we will denote by P_λ the orthogonal projector onto the corresponding eigenspace. Let $\mathrm{P}_\lambda(\varepsilon)$ stand for the orthogonal projector onto the finite dimensional space spanned by all eigenfunctions corresponding to those eigenvalues λ_ε of $\mathrm{H}_\varepsilon(\alpha, Q)$ that $\lambda_\varepsilon \to \lambda$ as $\varepsilon \to 0$. The results of [22] may be summed up in the following theorem.

Theorem 1.1 *All eigenvalues of* $\mathrm{H}_\varepsilon(\alpha, Q)$ *(except at most a finite number) are bounded as* $\varepsilon \to 0$. *Let* λ_ε *be an eigenvalue of* $\mathrm{H}_\varepsilon(\alpha, Q)$ *bounded as* $\varepsilon \to 0$, *then* λ_ε *has a finite limit* λ *that is a point of the spectrum of* $\mathrm{H}(\alpha, Q)$. *Moreover,* $\|\mathrm{P}_\lambda(\varepsilon) - \mathrm{P}_\lambda\| \to 0$ *as* $\varepsilon \to 0$. *Conversely, if* λ *is an eigenvalue of* $\mathrm{H}(\alpha, Q)$, *then there exists an eigenvalue* λ_ε *of* $\mathrm{H}_\varepsilon(\alpha, Q)$ *such that* $\lambda_\varepsilon \to \lambda$ *as* $\varepsilon \to 0$.

Although it has been showed the close proximity of the energy levels and pure states for the limiting and regularized Hamiltonians, we are still in the dark about convergence as $\varepsilon \to 0$ of the operators $\mathrm{H}_\varepsilon(\alpha, Q)$ in any topology. As a result, we do not know anything about convergence of the scattering quantities or other physical characteristics. Our objective in this paper is to give one further motivation for the choice of the Hamiltonian $\mathrm{H}(\alpha, Q)$. We shall study the scattering properties of the finite-range potentials αQ_ε on the graph Γ_ε in the limit $\varepsilon \to 0$. We prove that the scattering coefficients depend on the intensity α and the function Q in such a way that for all values of α the barrier αQ_ε is completely opaque except for the set Σ of resonant values, at which a partial transmission through the potential occurs. It will also be shown that the scattering amplitude for the Hamiltonians $\mathrm{H}_\varepsilon(\alpha, Q)$ and H_0 converges as $\varepsilon \to 0$ to that for the limiting Hamiltonian $\mathrm{H}(\alpha, Q)$. Here the Hamiltonian H_0 of a free particle on Γ acts via $\mathrm{H}_0 f = -f''$ on its domain consisting of those functions from $H^2(\Gamma)$ that are continuous on Γ and satisfy the Kirchhoff boundary conditions at the vertex.

2. Scattering problem for the δ′-like potential on the graph

Let us state the main result of this paper.

Theorem 2.1 *For each* $k > 0$ *and* $\alpha \in \mathbb{R}$ *the scattering matrix for the operators* $\mathrm{H}_\varepsilon(\alpha, Q)$ *and* H_0 *converge as* $\varepsilon \to 0$ *to the scattering matrix for* $\mathrm{H}(\alpha, Q)$ *and* H_0.

To start with, we briefly treat stationary scattering on the graph Γ associated with the Hamiltonians $\mathrm{H}(\alpha, Q)$ and H_0. From now on we assume that q is a zero function, i.e., the operator $\mathrm{H}(\alpha, Q)$ involves no potential and $\mathrm{H}_\varepsilon(\alpha, Q)$ has only perturbed potential αQ_ε. It is sufficient for us to look at the nontrivial case when the coupling constant α

belongs to the resonant set $\Sigma = \Sigma_1 \cup \Sigma_2$, since in the opposite case the particle is always reflected. Let us introduce a natural parametrization $s \in (0, +\infty)$ on each edge of Γ and Γ_ε, where $s = 0$ corresponds to the vertex a. Consider the incoming monochromatic wave e^{-iks} with $k > 0$ coming from infinity along the edge γ_n. The corresponding wavefunction has the form

$$\psi_n(s, k) = \begin{cases} T_{nm} e^{iks} & \text{on } \gamma_m, \ m \neq n, \\ e^{-iks} + T_{nn} e^{iks} & \text{on } \gamma_n. \end{cases} \tag{7}$$

Here T_{nn} are the reflection coefficients, and T_{nm} are the transmission coefficients. Substituting ψ_n into the matching conditions (5) or (6) we derive the scattering matrix that can be expressed via the coupling function

$$S(\alpha) = (-1)^{j-1} 2 \begin{pmatrix} \theta_1^2(\alpha) - \frac{1}{2} & \theta_1(\alpha)\theta_2(\alpha) & \theta_1(\alpha)\theta_3(\alpha) \\ \theta_1(\alpha)\theta_2(\alpha) & \theta_2^2(\alpha) - \frac{1}{2} & \theta_2(\alpha)\theta_3(\alpha) \\ \theta_1(\alpha)\theta_3(\alpha) & \theta_2(\alpha)\theta_3(\alpha) & \theta_3^2(\alpha) - \frac{1}{2} \end{pmatrix}$$

if $\alpha \in \Sigma_j$ for $j = 1, 2$. Note that the scattering matrix does not depend on k.

Next let us look in detail at stationary scattering for the Hamiltonians $H_\varepsilon(\alpha, Q)$ and H_0 and find the limit as $\varepsilon \to 0$ of the scattering amplitude. Consider the incoming monochromatic wave e^{-iks} coming from infinity along the edge γ_3.

We shall seek a positive-energy solution of the problem

$$-y'' + \alpha Q_\varepsilon y = k^2 y \quad \text{on } \Gamma \setminus \{a\}, \qquad y_{\omega_1}(a) = y_{\omega_2}(a) = y_{\omega_3}(a), \qquad \sum_{j=1}^{3} \frac{dy}{d\omega_j}(a) = 0 \tag{8}$$

that coincides on $\Gamma_\varepsilon \setminus \Omega_\varepsilon$ with ψ_3 given by (7). Let $u_\alpha = u_\alpha(\xi, \varkappa)$, $v_\alpha = v_\alpha(\xi, \varkappa)$ and $w_\alpha = w_\alpha(\xi, \varkappa)$ be a linearly independent system of solutions of the problem

$$-g'' + \alpha Q g = \varkappa^2 g \quad \text{on } \Omega \setminus \{b\}, \qquad g_{\omega_1}(b) = g_{\omega_2}(b) = g_{\omega_3}(b), \qquad \sum_{j=1}^{3} \frac{dg}{d\omega_j}(b) = 0 \tag{9}$$

on the graph Ω. Clearly, the functions $u_\alpha((x-a)/\varepsilon, \varepsilon k)$, $v_\alpha = v_\alpha((x-a)/\varepsilon, \varepsilon k)$ and $w_\alpha = w_\alpha((x-a)/\varepsilon, \varepsilon k)$ form a linearly independent system of solutions of the problem (8) on Ω_ε. Hence the desired solution on Ω_ε can be represented as a linear combination of this functions with the coefficients A, B and C respectively. Write $\varkappa = \varepsilon k$. Since the potential αQ_ε has discontinuities at the points a_1^ε, a_2^ε and a_3^ε, we demand that the conditions

$$y_{\gamma_j^\varepsilon}(a_j^\varepsilon) = y_{\omega_j^\varepsilon}(a_j^\varepsilon), \qquad \frac{dy}{d\gamma_j^\varepsilon}(a_j^\varepsilon) + \frac{dy}{d\omega_j^\varepsilon}(a_j^\varepsilon) = 0, \quad j = 1, 2, 3 \tag{10}$$

hold. On substituting our solution into (10) we obtain the linear system $Mc = r$ for the vector of unknown coefficients $c(\alpha, \varkappa) = (T_{31}, T_{32}, T_{33}, A, B, C)^\top$. Here $r(\varkappa) =$

$(0,0,0,0,e^{-i\varkappa},-i\varkappa e^{-i\varkappa})^\top$ and

$$
M(\alpha,\varkappa) = \begin{pmatrix}
-e^{i\varkappa} & 0 & 0 & u_\alpha(a_1,\varkappa) & v_\alpha(a_1,\varkappa) & w_\alpha(a_1,\varkappa) \\
-i\varkappa e^{i\varkappa} & 0 & 0 & \frac{du_\alpha}{d\omega_1}(a_1,\varkappa) & \frac{dv_\alpha}{d\omega_1}(a_1,\varkappa) & \frac{dw_\alpha}{d\omega_1}(a_1,\varkappa) \\
0 & -e^{i\varkappa} & 0 & u_\alpha(a_2,\varkappa) & v_\alpha(a_2,\varkappa) & w_\alpha(a_2,\varkappa) \\
0 & -i\varkappa e^{i\varkappa} & 0 & \frac{du_\alpha}{d\omega_2}(a_2,\varkappa) & \frac{dv_\alpha}{d\omega_2}(a_2,\varkappa) & \frac{dw_\alpha}{d\omega_2}(a_2,\varkappa) \\
0 & 0 & -e^{i\varkappa} & u_\alpha(a_3,\varkappa) & v_\alpha(a_3,\varkappa) & w_\alpha(a_3,\varkappa) \\
0 & 0 & -i\varkappa e^{i\varkappa} & \frac{du_\alpha}{d\omega_3}(a_3,\varkappa) & \frac{dv_\alpha}{d\omega_3}(a_3,\varkappa) & \frac{dw_\alpha}{d\omega_3}(a_3,\varkappa)
\end{pmatrix}.
$$

Clearly, the functions $u_\alpha(\cdot,0)$, $v_\alpha(\cdot,0)$ and $w_\alpha(\cdot,0)$ form a linearly independent system of solutions of the problem (2), (3). In what follows we shall omit the second argument of these functions and keep in mind that it equals zero. We introduce the determinants

$$
h_0(\alpha) = \begin{vmatrix}
u_\alpha(a_1) & v_\alpha(a_1) & w_\alpha(a_1) \\
u_\alpha(a_2) & v_\alpha(a_2) & w_\alpha(a_2) \\
u_\alpha(a_3) & v_\alpha(a_3) & w_\alpha(a_3)
\end{vmatrix}, \qquad
h_1(\alpha) = \begin{vmatrix}
\frac{du_\alpha}{d\omega_1}(a_1) & \frac{dv_\alpha}{d\omega_1}(a_1) & \frac{dw_\alpha}{d\omega_1}(a_1) \\
\frac{du_\alpha}{d\omega_2}(a_2) & \frac{dv_\alpha}{d\omega_2}(a_2) & \frac{dw_\alpha}{d\omega_2}(a_2) \\
\frac{du_\alpha}{d\omega_3}(a_3) & \frac{dv_\alpha}{d\omega_3}(a_3) & \frac{dw_\alpha}{d\omega_3}(a_3)
\end{vmatrix}.
$$

Let us also consider the function h_{nm}, which is just the determinant h_{1-n} with the mth-row replaced by the mth-row of h_n for $n=0,1$; $m=1,2,3$.

Let us denote by $\Delta(\varkappa,\alpha)$ the determinant of the matrix $M(\varkappa,\alpha)$. It admits the asymptotic expansion

$$\Delta(\varkappa,\alpha) = h_1(\alpha) + i\varkappa\{3h_1(\alpha) - H_0(\alpha)\} + \varkappa^2\{-9/2\,h_1(\alpha) + 3H_0(\alpha) - H_1(\alpha)\} + O(\varkappa^3)$$

as $\varkappa \to 0$, where $H_n(\alpha) = \sum_{m=1}^3 h_{nm}(\alpha)$ for $n=0,1$. Employing Cramer's rule, we derive

$$
T_{31}(\varkappa,\alpha) = \frac{-2i\varkappa(1+i\varkappa)}{\Delta(\varkappa,\alpha)} \begin{vmatrix}
u_\alpha(a_1) & v_\alpha(a_1) & w_\alpha(a_1) \\
\frac{du_\alpha}{d\omega_1}(a_1) & \frac{dv_\alpha}{d\omega_1}(a_1) & \frac{dw_\alpha}{d\omega_1}(a_1) \\
\frac{du_\alpha}{d\omega_2}(a_2) & \frac{dv_\alpha}{d\omega_2}(a_2) & \frac{dw_\alpha}{d\omega_2}(a_2)
\end{vmatrix}
$$
$$
\qquad\qquad -\frac{2\varkappa^2}{\Delta(\varkappa,\alpha)} \begin{vmatrix}
u_\alpha(a_1) & v_\alpha(a_1) & w_\alpha(a_1) \\
\frac{du_\alpha}{d\omega_1}(a_1) & \frac{dv_\alpha}{d\omega_1}(a_1) & \frac{dw_\alpha}{d\omega_1}(a_1) \\
u_\alpha(a_2) & v_\alpha(a_2) & w_\alpha(a_2)
\end{vmatrix} + O(\varkappa^3),
$$

$$
T_{32}(\varkappa,\alpha) = \frac{2i\varkappa(1+i\varkappa)}{\Delta(\varkappa,\alpha)} \begin{vmatrix}
\frac{du_\alpha}{d\omega_1}(a_1) & \frac{dv_\alpha}{d\omega_1}(a_1) & \frac{dw_\alpha}{d\omega_1}(a_1) \\
u_\alpha(a_2) & v_\alpha(a_2) & w_\alpha(a_2) \\
\frac{du_\alpha}{d\omega_2}(a_2) & \frac{dv_\alpha}{d\omega_2}(a_2) & \frac{dw_\alpha}{d\omega_2}(a_2)
\end{vmatrix}
$$
$$
\qquad\qquad +\frac{2\varkappa^2}{\Delta(\varkappa,\alpha)} \begin{vmatrix}
u_\alpha(a_1) & v_\alpha(a_1) & w_\alpha(a_1) \\
u_\alpha(a_2) & v_\alpha(a_2) & w_\alpha(a_2) \\
\frac{du_\alpha}{d\omega_2}(a_2) & \frac{dv_\alpha}{d\omega_2}(a_2) & \frac{dw_\alpha}{d\omega_2}(a_2)
\end{vmatrix} + O(\varkappa^3),
$$

$$
\begin{aligned}
T_{33}(\varkappa,\alpha) = \; &(\Delta(\varkappa,\alpha))^{-1}\Big[-h_1(\alpha) + i\varkappa\{H_0(\alpha) - 2h_{03}(\alpha) - h_1(\alpha)\} \\
&-\varkappa^2\{H_1(\alpha) + H_0(\alpha) - 2(h_{13}(\alpha) + h_{03}(\alpha)) - 1/2h_1(\alpha)\}\Big] + O(\varkappa^3)
\end{aligned}
$$

(11)

as $\varkappa \to 0$. Similar asymptotics can be derived without difficulty in the case, when the wave package comes from infinity along γ_1 or γ_2. We shall study the asymptotic behavior as $\varkappa \to 0$ of the scattering coefficients. Three cases are to be distinguished: $\Delta(\varkappa,\alpha) = O(1)$, $\Delta(\varkappa,\alpha) = O(\varkappa)$ and $\Delta(\varkappa,\alpha) = O(\varkappa^2)$ as $\varkappa \to 0$.

Lemma 2.2 *The set of roots of the equation $h_1(z) = 0$ coincides with the resonant set, i.e., h_1 is a characteristic determinant of the eigenvalue problem (2)–(4).*

Proof. It is easily seen that the system u_α, v_α and w_α of solutions of the problem (9) may be chosen so that

$$u_\alpha(b) = 0, \qquad \frac{du_\alpha}{d\omega_1}(b) = 0, \qquad \frac{du_\alpha}{d\omega_2}(b) = 1,$$

$$v_\alpha(b) = 0, \qquad \frac{dv_\alpha}{d\omega_2}(b) = 0, \qquad \frac{dv_\alpha}{d\omega_1}(b) = 1.$$

Clearly, u_α vanishes on ω_1, and v_α vanishes on ω_2. Let us consider the linear combination

$$g = \frac{dv_\alpha}{d\omega_1}(a_1)\frac{dw_\alpha}{d\omega_2}(a_2)u_\alpha + \frac{du_\alpha}{d\omega_2}(a_2)\frac{dw_\alpha}{d\omega_1}(a_1)v_\alpha - \frac{du_\alpha}{d\omega_2}(a_2)\frac{dv_\alpha}{d\omega_1}(a_1)w_\alpha. \quad (12)$$

By construction $\frac{dg}{d\omega_1}(a_1) = \frac{dg}{d\omega_2}(a_2) = 0$ and $\frac{dg}{d\omega_3}(a_3) = h_1(\alpha)$. Hence if $h_1(\alpha) = 0$ and g is nontrivial on Ω, then g is an eigenfunction of the problem (2), corresponding to α, and therefore $\alpha \in \Sigma$.

Note that at least one of the values $\frac{du_\alpha}{d\omega_j}(a_j)$, $\frac{dv_\alpha}{d\omega_j}(a_j)$ or $\frac{dw_\alpha}{d\omega_j}(a_j)$ is different from zero for $j = 1, 2, 3$. Now consider the exceptional case, when g is trivial, i.e., all coefficients in (12) equal zero. We show then that both values $\frac{du_\alpha}{d\omega_2}(a_2)$ and $\frac{dv_\alpha}{d\omega_1}(a_1)$ are zero. Conversely, suppose that $\frac{dv_\alpha}{d\omega_1}(a_1) \neq 0$. Since $\frac{dv_\alpha}{d\omega_1}(a_1)\frac{dw_\alpha}{d\omega_2}(a_2) = 0$, one obtains $\frac{du_\alpha}{d\omega_2}(a_2) = \frac{dv_\alpha}{d\omega_2}(a_2) = \frac{dw_\alpha}{d\omega_2}(a_2) = 0$, a contradiction. Similarly, $\frac{du_\alpha}{d\omega_2}(a_2) = 0$. It follows that $\frac{du_\alpha}{d\omega_1}(a_1) = \frac{du_\alpha}{d\omega_2}(a_2) = 0$ and $\frac{dv_\alpha}{d\omega_1}(a_1) = \frac{dv_\alpha}{d\omega_2}(a_2) = 0$. Thus the function $g_1 = \frac{dv_\alpha}{d\omega_3}(a_3)u_\alpha - \frac{du_\alpha}{d\omega_3}(a_3)v_\alpha$ satisfies $\frac{dg_1}{d\omega_1}(a_1) = \frac{dg_1}{d\omega_2}(a_2) = \frac{dg_1}{d\omega_3}(a_3) = 0$. If g_1 is nontrivial on Ω, then it is an eigenfunction of the problem (2), corresponding to α. If not, then u_α would be a desired eigenfunction.

What is left is to show that $\alpha \in \Sigma$ implies $h_1(\alpha) = 0$. If we choose the new base u_α, v_α and w_α such that u_α is an eigenfunction of the problem (2)–(4) corresponding to α, then the first column of the determinant is zero and the assertion follows. $\qquad \square$

Lemma 2.3 *The set of roots of the equation $H_0(z) = 0$ that belongs to the resonant set coincides with Σ_2.*

Proof. Suppose, contrary to our claim, that there exists $\alpha \in \Sigma_1$ such that $H_0(\alpha) = 0$. We choose the new linearly independent system u_α, v_α and w_α such that u_α is an eigenfunction of the problem (2)–(4) corresponding to α. Write $V = (\frac{dv_\alpha}{d\omega_1}(a_1), \frac{dv_\alpha}{d\omega_2}(a_2), \frac{dv_\alpha}{d\omega_3}(a_3))$ and $W = (\frac{dw_\alpha}{d\omega_1}(a_1), \frac{dw_\alpha}{d\omega_2}(a_2), \frac{dw_\alpha}{d\omega_3}(a_3))$. These vectors are linearly independent. Indeed, in the opposite case there exists a nonzero number λ such that $V = \lambda W$. Therefore $v_\alpha - \lambda w_\alpha$ is an eigenfunction of the problem (2)–(4) corresponding to α that is linearly independent with u_α, a contradiction.

In view of the Lagrange identities

$$u_\alpha(a_1)\frac{dv_\alpha}{d\omega_1}(a_1) + u_\alpha(a_2)\frac{dv_\alpha}{d\omega_2}(a_2) + u_\alpha(a_3)\frac{dv_\alpha}{d\omega_3}(a_3) = 0,$$

$$u_\alpha(a_1)\frac{dw_\alpha}{d\omega_1}(a_1) + u_\alpha(a_2)\frac{dw_\alpha}{d\omega_2}(a_2) + u_\alpha(a_3)\frac{dw_\alpha}{d\omega_3}(a_3) = 0$$

the vector $U = (u_\alpha(a_1), u_\alpha(a_2), u_\alpha(a_3))$ is orthogonal to both vectors V and W. The identity $H_0(\alpha) = 0$ can be written as

$$u_\alpha(a_1) \begin{vmatrix} \frac{dv_\alpha}{d\omega_2}(a_2) & \frac{dv_\alpha}{d\omega_3}(a_3) \\ \frac{dw_\alpha}{d\omega_2}(a_2) & \frac{dw_\alpha}{d\omega_3}(a_3) \end{vmatrix} + u_\alpha(a_2) \begin{vmatrix} \frac{dv_\alpha}{d\omega_3}(a_3) & \frac{dv_\alpha}{d\omega_1}(a_1) \\ \frac{dw_\alpha}{d\omega_3}(a_3) & \frac{dw_\alpha}{d\omega_1}(a_1) \end{vmatrix}$$

$$+ u_\alpha(a_3) \begin{vmatrix} \frac{dv_\alpha}{d\omega_1}(a_1) & \frac{dv_\alpha}{d\omega_2}(a_2) \\ \frac{dw_\alpha}{d\omega_1}(a_1) & \frac{dw_\alpha}{d\omega_2}(a_2) \end{vmatrix} = 0.$$

It follows that U is orthogonal to the vector product $V \times W$, hence that U is a zero vector, i.e., $u_\alpha(a_1) = u_\alpha(a_2) = u_\alpha(a_3) = 0$, and finally that u_α is a zero function, which is impossible.

Let α belong to Σ_2. We select a linearly independent system u_α, v_α and w_α such that u_α and v_α form a base in the corresponding eigenspace. Obviously, the determinant $H_0(\alpha)$ vanishes, and the proof is complete. $\qquad\square$

Remark 2.4 *The roots of the equation $h_1(z) = 0$ as well as $H_0(z) = 0$ do not depend on the choice of the linearly independent system of solutions of the problem (9).*

Lemma 2.5 *The function H_1 is different from zero on Σ_2.*

Proof. On the contrary, suppose that there exists $\alpha \in \Sigma_2$ such that $H_1(\alpha) = 0$. Let us select the system u_α, v_α and w_α of linearly independent solutions of the problem (9) such that u_α and v_α are eigenfunctions of the problem (2)–(4) corresponding to α. Set $U = (u_\alpha(a_1), u_\alpha(a_2), u_\alpha(a_3))$ and $V = (v_\alpha(a_1), v_\alpha(a_2), v_\alpha(a_3))$. Note that the vectors U and V are linearly independent, since in the opposite case the functions u_α and v_α would be linearly dependent.

From the Lagrange identities

$$u_\alpha(a_1)\frac{dw_\alpha}{d\omega_1}(a_1) + u_\alpha(a_2)\frac{dw_\alpha}{d\omega_2}(a_2) + u_\alpha(a_3)\frac{dw_\alpha}{d\omega_3}(a_3) = 0$$

$$v_\alpha(a_1)\frac{dw_\alpha}{d\omega_1}(a_1) + v_\alpha(a_2)\frac{dw_\alpha}{d\omega_2}(a_2) + v_\alpha(a_3)\frac{dw_\alpha}{d\omega_3}(a_3) = 0$$

it follows that the vector $W = (\frac{dw_\alpha}{d\omega_1}(a_1), \frac{dw_\alpha}{d\omega_2}(a_2), \frac{dw_\alpha}{d\omega_3}(a_3))$ is orthogonal to U and V. The identity $H_1(\alpha) = 0$ can be written in the form

$$\frac{dw_\alpha}{d\omega_1}(a_1) \begin{vmatrix} u_\alpha(a_2) & u_\alpha(a_3) \\ v_\alpha(a_2) & v_\alpha(a_3) \end{vmatrix} + \frac{dw_\alpha}{d\omega_2}(a_2) \begin{vmatrix} u_\alpha(a_3) & u_\alpha(a_1) \\ v_\alpha(a_3) & v_\alpha(a_1) \end{vmatrix}$$

$$+ \frac{dw_\alpha}{d\omega_3}(a_3) \begin{vmatrix} u_\alpha(a_1) & u_\alpha(a_2) \\ v_\alpha(a_1) & v_\alpha(a_2) \end{vmatrix} = 0.$$

Consequently, W is orthogonal to the vector product $U \times V$. Analysis similar to that in the proof of the previous lemma gives the assertion of the lemma. $\qquad\square$

We are now in a position to prove the main result. Without loss of generality we establish the convergence of the transmission coefficient $T_{31}(\varepsilon k, \alpha)$ as $\varepsilon \to 0$. Other coefficients may be handled in much the same way.

Proof of Theorem 2.1. *The non-resonant case.* Since α is not a resonant coupling constant, $h_1(\alpha) \neq 0$. From (11) it immediately follows that $T_{31}(\varepsilon k, \alpha) = O(\varepsilon k)$ as $\varepsilon \to 0$.

The case of simple resonance. If $\alpha \in \Sigma_1$, then $h_1(\alpha) = 0$ and $H_0(\alpha) \neq 0$ in light of Lemmas 2.2, 2.3. By (11)

$$T_{31}(\varepsilon k, \alpha) = \frac{2}{H_0(\alpha)} \begin{vmatrix} u_\alpha(a_1) & v_\alpha(a_1) & w_\alpha(a_1) \\ \frac{du_\alpha}{d\omega_1}(a_1) & \frac{dv_\alpha}{d\omega_1}(a_1) & \frac{dw_\alpha}{d\omega_1}(a_1) \\ \frac{du_\alpha}{d\omega_2}(a_2) & \frac{dv_\alpha}{d\omega_2}(a_2) & \frac{dw_\alpha}{d\omega_2}(a_2) \end{vmatrix} + O(\varepsilon k), \qquad \varepsilon \to 0.$$

We continue by choosing the new linearly independent system u_α, v_α and w_α such that u_α is an eigenfunction of the problem (2)–(4) corresponding to α satisfying the condition $u_\alpha^2(a_1) + u_\alpha^2(a_2) + u_\alpha^2(a_3) = 1$ and $\frac{dv_\alpha}{d\omega_2}(a_2) = 0$. Observe that

$$T_{31}(0, \alpha) = 2\theta_1(\alpha) \frac{dv_\alpha}{d\omega_1}(a_1) \frac{dw_\alpha}{d\omega_2}(a_2) H_0^{-1}(\alpha). \tag{13}$$

Now let us express $T_{31}(0, \alpha)$ via the coupling function. Taking into account the Lagrange identities we see that

$$\frac{u_\alpha(a_1)}{u_\alpha(a_3)} = -\frac{\frac{dv_\alpha}{d\omega_3}(a_3)}{\frac{dv_\alpha}{d\omega_1}(a_1)}, \qquad \frac{u_\alpha(a_2)}{u_\alpha(a_3)} = \frac{\frac{dv_\alpha}{d\omega_3}(a_3)\frac{dw_\alpha}{d\omega_1}(a_1) - \frac{dv_\alpha}{d\omega_1}(a_1)\frac{dw_\alpha}{d\omega_3}(a_3)}{\frac{dv_\alpha}{d\omega_1}(a_1)\frac{dw_\alpha}{d\omega_2}(a_2)}.$$

Therefore

$$h_{01}(\alpha) = -u_\alpha(a_1)\frac{dv_\alpha}{d\omega_3}(a_3)\frac{dw_\alpha}{d\omega_2}(a_2) = \frac{u_\alpha^2(a_1)\frac{dv_\alpha}{d\omega_1}(a_1)\frac{dw_\alpha}{d\omega_2}(a_2)}{\theta_3(\alpha)}, \tag{14}$$

$$h_{02}(\alpha) = u_\alpha(a_2)\left\{ \frac{dv_\alpha}{d\omega_3}(a_3)\frac{dw_\alpha}{d\omega_1}(a_1) - \frac{dv_\alpha}{d\omega_1}(a_1)\frac{dw_\alpha}{d\omega_3}(a_3) \right\} = \frac{u_\alpha^2(a_2)\frac{dv_\alpha}{d\omega_1}(a_1)\frac{dw_\alpha}{d\omega_2}(a_2)}{\theta_3(\alpha)}. \tag{15}$$

By construction $h_{03}(\alpha) = u_\alpha^2(a_3)\frac{dv_\alpha}{d\omega_1}(a_1)\frac{dw_\alpha}{d\omega_2}(a_2)/\theta_3(\alpha)$. Recall that $H_0(\alpha) = h_{01}(\alpha) + h_{02}(\alpha) + h_{03}(\alpha)$. Combining (14) and (15) yields

$$H_0 = \frac{\frac{dv_\alpha}{d\omega_1}(a_1)\frac{dw_\alpha}{d\omega_2}(a_2)}{\theta_3(\alpha)}\{u_\alpha^2(a_1) + u_\alpha^2(a_2) + u_\alpha^2(a_3)\} = \frac{\frac{dv_\alpha}{d\omega_1}(a_1)\frac{dw_\alpha}{d\omega_2}(a_2)}{\theta_3(\alpha)}.$$

Then by (13)

$$T_{31}(0, \alpha) = 2\theta_1(\alpha)\theta_3(\alpha) = T_{31}(\alpha).$$

The case of double resonance. Finally suppose that $\alpha \in \Sigma_2$. By Lemmas 2.2, 2.3 and 2.5 we have $h_1(\alpha) = H_0(\alpha) = 0$ and $H_1(\alpha) \neq 0$, thus

$$T_{31}(\varepsilon k, \alpha) = \frac{2}{H_1(\alpha)} \begin{vmatrix} u_\alpha(a_1) & v_\alpha(a_1) & w_\alpha(a_1) \\ \frac{du_\alpha}{d\omega_1}(a_1) & \frac{dv_\alpha}{d\omega_1}(a_1) & \frac{dw_\alpha}{d\omega_1}(a_1) \\ u_\alpha(a_2) & v_\alpha(a_2) & w_\alpha(a_2) \end{vmatrix} + O(\varepsilon k), \qquad \varepsilon \to 0$$

by (11). Let us choose the new base u_α, v_α and w_α such that u_α and v_α are eigenfunctions of the problem (2) corresponding to α satisfying $\theta_1^2(\alpha) + \theta_2^2(\alpha) + \theta_3^2(\alpha) = 1$. Consequently,

$$T_{31}(0, \alpha) = -2\theta_3(\alpha)\frac{dw_\alpha}{d\omega_1}(a_1)H_1^{-1}(\alpha).$$

Since the Lagrange identities hold it follows that

$$\frac{\frac{dw_\alpha}{d\omega_1}(a_1)}{\theta_1(\alpha)} = \frac{\frac{dw_\alpha}{d\omega_2}(a_2)}{\theta_2(\alpha)} = \frac{\frac{dw_\alpha}{d\omega_3}(a_3)}{\theta_3(\alpha)}.$$

We thus get

$$h_{11}(\alpha) = \theta_1(\alpha)\frac{dw_\alpha}{d\omega_1}(a_1), \qquad h_{12}(\alpha) = \theta_2(\alpha)\frac{dw_\alpha}{d\omega_2}(a_2) = \frac{\theta_2^2(\alpha)}{\theta_1(\alpha)}\frac{dw_\alpha}{d\omega_1}(a_1),$$

$$h_{13}(\alpha) = \theta_3(\alpha)\frac{dw_\alpha}{d\omega_3}(a_3) = \frac{\theta_3^2(\alpha)}{\theta_1(\alpha)}\frac{dw_\alpha}{d\omega_1}(a_1).$$

By recalling $H_1(\alpha) = h_{11}(\alpha) + h_{12}(\alpha) + h_{13}(\alpha)$, one obtains

$$H_1(\alpha) = \frac{\frac{dw_\alpha}{d\omega_1}(a_1)}{\theta_1(\alpha)}\{\theta_1^2(\alpha) + \theta_2^2(\alpha) + \theta_3^2(\alpha)\} = \frac{\frac{dw_\alpha}{d\omega_1}(a_1)}{\theta_1(\alpha)},$$

hence

$$T_{31}(0,\alpha) = -2\theta_1(\alpha)\theta_3(\alpha) = T_{31}(\alpha),$$

and the proof is complete. □

3. An example

We suppose that the shape of a short-range potential in an actual model can be approximately described as follows

$$Q = \begin{cases} 7 & \text{on } \omega_1,\ s \in [0, 0.5), \\ -7 & \text{on } \omega_1,\ s \in (0.5, 1], \\ 0 & \text{otherwise.} \end{cases}$$

We study the scattering properties of the potentials $\alpha\varepsilon^{-2}Q((x-a)/\varepsilon)$ on Γ_ε as $\varepsilon \to 0$ for these models. As shown before, the scattering amplitude for the Hamiltonians $H_\varepsilon(\alpha, Q)$ and H_0 converges as $\varepsilon \to 0$ to that for the limiting Hamiltonian $H(\alpha, Q)$. Certainly, the resonant set and the coupling function to be found are specific to the given shape Q.

Table 1. Resonant intensities and coupling function

α	$\theta_1(\alpha)$	$\theta_2(\alpha) = \theta_3(\alpha)$
8.8104	−0.9992	0.0279
28.5513	0.9999	0.0012
59.5701	−0.9999	0.997×10^{-4}

Since the function Q vanishes outside the edge ω_1, the resonant set Σ coincides with the set Σ_1 of the simple eigenvalues of (2)–(4). In fact, all eigenfunctions are constant outside the edge ω_1 and the resonant set is the spectrum of the problem

$$-g'' + \alpha Q g = 0 \quad \text{on} \quad \omega_1, \qquad \frac{dg}{d\omega_1}(a_1) = \frac{dg}{d\omega_1}(b) = 0.$$

Table 2. Scattering matrix for $\alpha = 8.8104$

0.9968	−0.0558	−0.0558
−0.0558	−0.9984	0.0016
−0.0558	0.0016	−0.9984

Table 3. Scattering matrix for $\alpha = 28.5513$
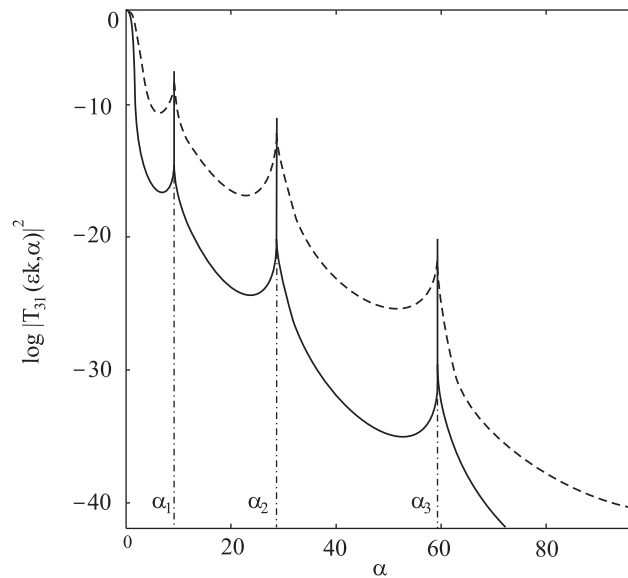
0.9996	0.0024	0.0024
0.0024	−0.9999	0.288×10^{-5}
0.0024	0.288×10^{-5}	−0.9999

Table 4. Scattering matrix for $\alpha = 59.5701$

0.9996	−0.0002	−0.0002
−0.0002	−0.9999	0.199×10^{-7}
−0.0002	0.199×10^{-7}	−0.9999

Table 1 lists the first three positive resonant values of α (numerically computed using *Maple*) and the corresponding values of the coupling function $\theta(\alpha)$. Tables 2, 3 and 4 list the scattering matrixes for different resonant intensities. We note that the transmission coefficients decay very fast.

The similar effects were observed, when the asymptotic behavior of the scattering coefficients for $H_\varepsilon(\alpha, Q)$ was discovered directly to the given shape Q. Figure 2 shows the logarithm of the transmission probability $|T_{31}(\varepsilon k, \alpha)|^2$ versus α for $\varepsilon k = 0.0001$ and

Figure 2. Logarithm of the transmission probability $|T_{31}(\varepsilon k, \alpha)|^2$ against α at $\varepsilon k = 0.0001$ (solid curve) and 0.01 (dash curve)

$\varepsilon k = 0.01$. From this figure we see that $|T_{31}(\varepsilon k, \alpha)|^2$ vanishes as $\varepsilon \to 0$ except when α takes resonant values which correspond to the spikes of $|T_{31}(\varepsilon k, \alpha)|^2$. Observe also that the transmission coefficient $T_{31}(\varepsilon k, \alpha)$ decays as $\alpha \to \infty$.

Acknowledgments

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